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THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE LAYER COIL—THE LORENZ APPARATUS AND THE AYRTON-JONES ABSOLUTE ELECTRODYNAMOMETER.

By Edward B. Rosa.

In the Lorenz experiment for determining resistance in absolute measure a circular disk is rotated in a magnetic field produced by an electric current in coaxial coils or in a coaxial helix, and the electromotive force induced in the disk is balanced against the difference of potential at the terminals of the resistance to be measured. This induced electromotive force is proportional to the speed and to the number of lines of magnetic force passing through the disk due to the current in the surrounding coils or helix. The latter usually consists of a single layer winding on a carefully ground cylindrical surface or in a very true screw thread cut in such a surface. In order to know the number of lines of magnetic force passing through the disk it is necessary to calculate the mutual inductance of the cylindrical winding and the circle forming the boundary of the disk.

1. LORENZ'S FORMULA.

This problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.[1] Assuming that the current was uniformly distributed over the surface of the cylinder forming a current sheet he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however,

[1] Wied. Annalen, **25**, p. 1; 1885.

the constant of the apparatus instead of mutual inductance and denoted it by C; it is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

$$C=\frac{\pi q r^2}{d}\left[Q(a_1)+Q(a_2)\right] \tag{1}$$

$$Q(a)=2\pi q\sqrt{\frac{a-1}{a}}\left[1+\frac{3}{8}\frac{q^2}{a^2}+\frac{5}{16}\frac{q^4}{a^4}\left(\frac{7}{4}-a\right)+\frac{35}{128}\frac{q^6}{a^6}\left(\frac{33}{8}-\frac{9}{2}a+a^2\right)+\ldots\right] \tag{2}$$

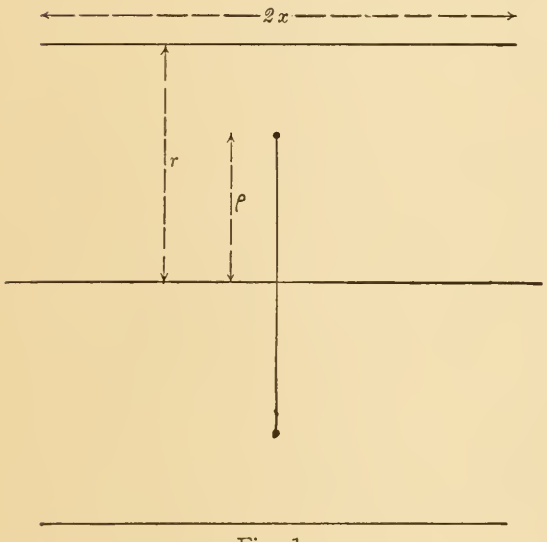

Fig. 1.

ρ = radius of the disk, Fig. 1.

r = radius of the solenoid.

$2x$ = length of winding of solenoid.

$q=\rho\, r$ = ratio of the two radii.

$d=\frac{2x}{n}$ = distance between centers of successive turns of wire.

$a=\frac{x^2+r^2}{r^2}$

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end of the solenoid and x_2 to the other,

$$a_1=\frac{x_1^2+r^2}{r^2} \qquad a_2=\frac{x_2^2+r^2}{r^2}$$

Then $Q(a_1)$ is found by substituting the values of a_1 in equation (2) above, and $Q(a_2)$ by using the value of a_2 for a in the same equation. The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant C, that is, the mutual inductance sought.

As Lorenz gave the expression for the general term of (2), his equation can be extended. The following is the general term:

$$Q(a)=2\pi\sum_{m=0}^{m=\infty}q^{2m+1}\frac{1.3\ldots.2m-1}{2.4\ldots.2m}\cdot\frac{1}{1.2\ldots.(m+1)}\cdot\frac{d^m}{da^m}\left(\frac{a-1}{a}\right)^{m+\frac{1}{2}} \quad (2a)$$

2. JONES'S FORMULÆ.

Two solutions were given by Jones,[2] both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when AO_1, Fig. 2, is less than the difference in the radii of inner and outer coils; that is, when AO_1 is less than $A-a$. As this is a serious limitation, and the formula is a laborious one to use, I shall not give it. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

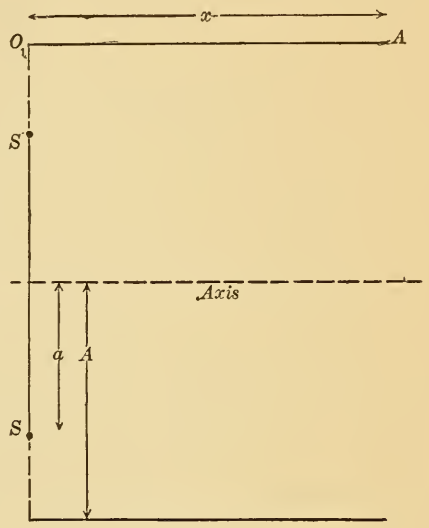

Fig. 2.

$$M_\theta=\Theta\,(A+a)\,ck\left\{\frac{F-E}{k^2}+\frac{c'^2}{c^2}\,(F-\Pi)\right\} \quad (3)$$

M_θ = mutual inductance of helix O_1A, Fig. 2, on the disk S in the plane of one end.

$\Theta=2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F and K are complete elliptic integrals to modulus k, where

$$k^2=\frac{4Aa}{(A+a)^2+x^2} \qquad c^2=\frac{4Aa}{(A+a)^2} \qquad c'^2=1-c^2.$$

Π = complete elliptic integral of the third kind, to modulus k.

The elliptic integral Π of the third kind can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables. The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

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[2] Proc. Roy. Soc., **63**, p. 198; 1898.

3. DERIVATION OF A NEW FORMULA.

I have derived an independent expression for the mutual inductance of a circle and a coaxial single-layer coil in the form of an algebraic series which involves no elliptic integrals, is relatively easy to calculate, and is very accurate. It is also better adapted for quick approximate calculations than Jones's formula. The formula

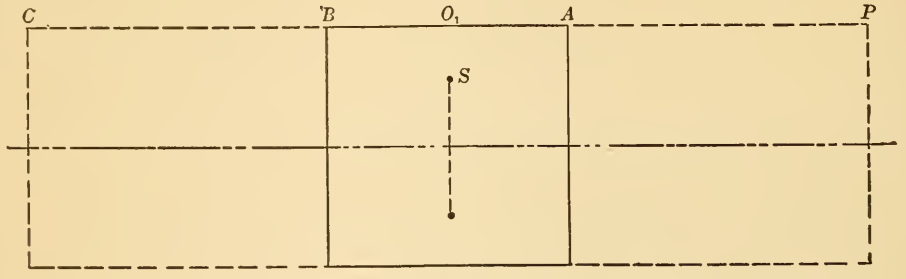

Fig. 3.

is based on the assumption that the current in the coil AB, Fig. 3, is a current sheet or is equivalent to a current sheet. The demonstration of Jones and Rayleigh that this is a legitimate assumption applies to an ideal winding. I shall discuss below the question whether in an actual instrument there is any appreciable error when this equivalence of a spiral winding and current sheet is assumed. If the outer solenoid were of infinite length the force within would be uniform and equal to $4\pi n_1 i_1$, where n_1 is the number of turns per unit of length in the solenoid and i_1 is the current in the solenoid. The area of the circle S, Fig. 3, being πa^2, the number of lines passing through S is $4\pi^2 a^2 n_1 i_1$, and therefore the mutual inductance of S and the infinite solenoid is $4\pi^2 a^2 n_1$, or for half the infinite solenoid O_1P it is

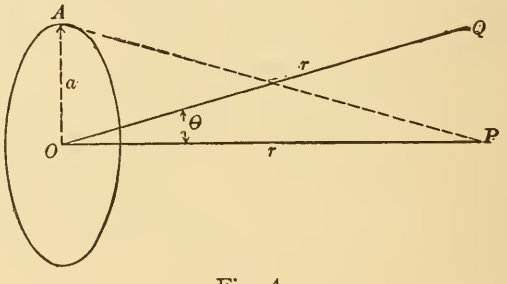

Fig. 4.

$$\tfrac{1}{2}M_\infty = 2\pi^2 a^2 n_1$$

If we find the part of this due to the end AP (shown dotted and extending to infinity) and subtract it, the remainder is the part due to O_1A, which is the quantity sought.

If OA, Fig. 4, is a disk of radius a and density n its potential V at P is

$$V_P = 2\pi n\left\{(a^2+r^2)^{\frac{1}{2}} - r\right\} = 2\pi n\left\{\frac{a^2}{2r} - \frac{1.1}{2.4}\frac{a^4}{r^3} + \frac{1.1.3}{2.4.6}\frac{a^6}{r^5} - \frac{1.1.3.5}{2.4.6.8}\frac{a^8}{r^7} + \ldots\right\} \quad (4)$$

For a point Q off the axis we must insert in (4) the zonal harmonics corresponding to the angle θ. Thus

$$V_Q = 2\pi n\left\{\frac{a^2}{2r}P_0 - \frac{1}{8}\frac{a^4}{r^3}P_2 + \frac{1}{16}\frac{a^6}{r^5}P_4 - \frac{5}{128}\frac{a^8}{r^7}P_6 + \frac{35}{1280}\frac{a^{10}}{r^9}P_8 - \ldots\right\} \quad (5)$$

V_Q will be the magnetic potential at any point Q if the disk is covered with a layer of positive magnetism of density n, or if it is the end of an infinite solenoid wound with n turns of wire per centimeter and carrying unit current. The expression is convergent when $r > a$. Differentiating V_Q with respect to r we have the magnetic force at Q in the direction of the vector r. Thus,

$$-\frac{dV_Q}{dr} = 2\pi n\left\{\frac{a^2}{2r^2}P_0 - \frac{3}{8}\frac{a^4}{r^4}P_2 + \frac{5}{16}\frac{a^6}{r^6}P_4 - \frac{35}{128}\frac{a^8}{r^8}P_6 + \frac{315}{1280}\frac{a^{10}}{r^{10}}P_8 - \ldots\right\} \quad (6)$$

The values of the zonal harmonics are as follows ($\mu = \cos\theta$):

$$P_0 = 1$$

$$P_2 = \frac{1}{2}(3\mu^2 - 1)$$

$$P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$

$$P_6 = \frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)$$

$$P_8 = \frac{1}{128}(6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35)$$

$$P_{10} = \frac{1}{256}(46189\mu^{10} - 109395\mu^8 + 90090\mu^6 - 30030\mu^4 + 3465\mu^2 - 63)$$

$$P_{12} = \frac{1}{1024}(52003\times 13\mu^{12} - 176358\times 11\mu^{10} + 230945\times 9\mu^8 - 145860\times 7\mu^6 + 45045\times 5\mu^4 - 6006\times 3\mu^2 + 231)$$

Substituting the above values of the zonal harmonics in (6) would give the value of the magnetic force at any point Q in terms of a, r, and μ.

Let R, Fig. 5, be a circle coaxial with S and lying on the spherical surface through Q with center O. A zone at Q on the spherical surface of angular width $d\theta$ and radius $r \sin \theta$ has an area $2\pi r^2 \sin \theta \, d\theta$ and through it pass dN lines of magnetic force.

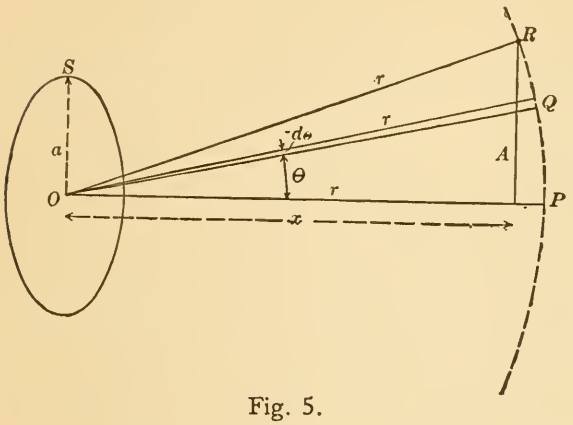

Fig. 5.

$$dN = 2\pi r^2 \sin \theta \, d\theta \left(\frac{dV_Q}{dr}\right)$$

Since $\mu = \cos \theta$,

$$d\mu = -\sin \theta \, d\theta$$

$$\therefore \; dN = 2\pi r^2 d\mu \left(-\frac{dV_Q}{dr}\right)$$

Or,
$$dN = 4\pi^2 n d\mu \left\{\frac{a^2}{2} - \frac{3}{16}\frac{a^4}{r^2}(3\mu^2 - 1) + \frac{5}{128}\frac{a^6}{r^4}(35\mu^4 - 30\mu^2 + 3) - \frac{35}{2048}\frac{a^8}{r^6}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5) + \ldots\right\} \qquad (7)$$

The whole number of lines of magnetic force N passing through the circle R of radius A and distance x from the magnetic disk S ($x = \sqrt{r^2 - A^2}$) is found by integrating (7) with respect to μ from $\mu = \frac{x}{r} = \frac{x}{\sqrt{A^2 + x^2}}$ to $\mu = 1$

Thus,

$$N = 2\pi^2 a^2 n \int_{\mu=\frac{x}{r}}^{\mu=1} \left\{1 - \frac{3}{8}\frac{a^2}{r^2}(3\mu^2 - 1) + \frac{5}{64}\frac{a^4}{r^4}(35\mu^4 - 30\mu^2 + 3) - \frac{35}{1024}\frac{a^6}{r^6}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5) + \ldots\right\} d\mu \qquad (8)$$

or

$$N=2\pi^2a^2n\left[\left(1-\frac{x}{(x^2+A^2)^{\frac{1}{2}}}\right)+\frac{3}{8}\frac{a^2}{(x^2+A^2)}\left(\frac{x^3}{(x^2+A^2)^{\frac{3}{2}}}-\frac{x}{(x^2+A^2)^{\frac{1}{2}}}\right)\right.$$
$$-\frac{5}{64}\frac{a^4}{(x^2+A^2)^2}\left(\frac{7x^5}{(x^2+A^2)^{\frac{5}{2}}}-\frac{10x^3}{(x^2+A^2)^{\frac{3}{2}}}+\frac{3x}{(x^2+A^2)^{\frac{1}{2}}}\right)$$
$$\left.+\frac{35}{128}\frac{a^6}{(x^2+A^2)^3}\left(\qquad\qquad\right)+\ldots\right] \qquad (9)$$

N is the number of lines of magnetic force due to the disk S (of radius a) passing through the coaxial circle R (of radius A) at distance x, Fig. 5. It is also the number of lines due to a larger disk of radius A, Fig. 6, passing through a circle of radius a at a

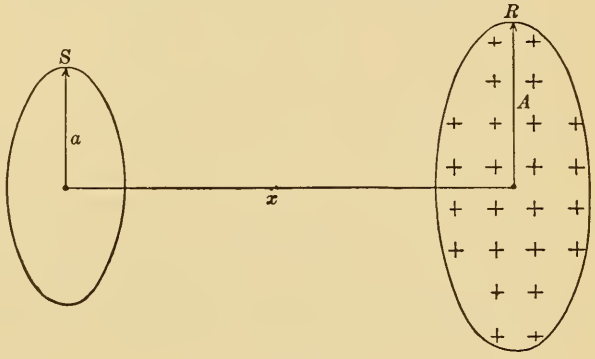

Fig. 6.

distance x; or the number of lines due to the semi-infinite solenoid AP, Fig. 3, wound with n turns per cm and carrying unit current, passing through the circle S. N is therefore the mutual inductance M_{AP} of the semi-infinite end AP, Fig. 3, which we are to subtract from $\frac{1}{2}M_\infty$ to give M the mutual inductance of the solenoid O_1A with respect to the circle S. Putting $\sqrt{x^2+A^2}=d$, equation (9) above reduces to

$$M_{AP}=2\pi^2a^2n\left[1-\frac{x}{d}-\frac{3}{8}\frac{a^2}{d^2}\frac{xA^2}{d^3}-\frac{5}{64}\frac{a^4}{d^4}\left(\frac{3xA^4-4x^3A^2}{d^5}\right)-\ldots\right] \qquad (10)$$

$$\therefore M_{O_1A}=\frac{1}{2}M_\infty-M_{AP}=\frac{2\pi^2a^2nx}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}\left(3-\frac{4x^2}{A^2}\right)\right.$$
$$\left.+\frac{35}{128}\frac{a^6A^6}{d^{12}}\left(\quad\right)+\ldots\right] \qquad (11)$$

This is the value of the mutual inductance of half the solenoid (O_1A) on the circle S sought. In order to obtain an expression for M that will give accurate numerical values when the circle S is relatively large it is necessary to carry out the series to a larger number of terms than have been given above. Substituting in (6) the values of all the zonal harmonics to P_{12}, integrating and reducing as is done above for the first terms, the following expression is obtained, which is amply accurate for the most refined experimental work. ($N=2n_1x$, the whole number of turns of wire on the solenoid in the length $OA=2x$).

$$M_{OA}=\frac{2\pi^2a^2N}{d}\left[1+\frac{3}{8}\frac{a^2A^2}{d^4}+\frac{5}{64}\frac{a^4A^4}{d^8}X_2+\frac{35}{512}\frac{a^6A^6}{d^{12}}X_4+\frac{63}{1024}\frac{a^8A^8}{d^{16}}X_6 + \frac{231}{4096}\frac{a^{10}A^{10}}{d^{20}}X_8+\frac{429}{16384}\frac{a^{12}A^{12}}{d^{24}}X_{10}+\ldots\right] \quad (12)$$

$$\left.\begin{aligned} X_2&=3-4\frac{x^2}{A^2}\\ X_4&=\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}\\ X_6&=\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}\\ X_8&=\frac{63}{32}-\frac{105}{4}\frac{x^2}{A^2}+63\frac{x^4}{A^4}-36\frac{x^6}{A^6}+4\frac{x^8}{A^8}\\ X_{10}&=\frac{231}{128}-\frac{1155}{32}\frac{x^2}{A^2}+\frac{1155}{8}\frac{x^4}{A^4}-165\frac{x^6}{A^6}+55\frac{x^8}{A^8}-4\frac{x^{10}}{A^{10}} \end{aligned}\right\} \quad (13)$$

a = radius of disk or circle S, Fig. 2.
A = radius of the solenoid.
x = length O_1A of one end of the solenoid.
$d=\sqrt{x^2+A^2}$ = half the diagonal of the solenoid.
N is the whole number of turns of wire in the length x.

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2A^2}{d^4}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corre-

sponding factor in each of the other terms. Similary, the various terms X_2, X_4, etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedious interpolations for incomplete elliptic integrals.

4. TESTS OF THE NEW FORMULA.

EXAMPLE 1.

As an example take the case given by Jones[3], to calculate the mutual inductance of the disk and solenoid of the Lorenz apparatus built for McGill University.

$A =$ 10.513365 inches.
$a =$ 6.509985 "
$x =$ 2.51240 "
$N =$ 201 turns.

The mutual inductance was computed by Mr. Rhodes under the direction of Professor Ayrton by Jones's first method. The dimensions in inches were used and the result then converted into centimeters. The value of M found was as follows:

$$M = 2M_\theta = 18056.36 \text{ inches} = 45862.33 \text{ cm}$$

After rewinding the coil and regrinding the disk the dimensions were as follows:

$A =$ 10.512295 inches.
$a =$ 6.507495 "
$x =$ 2.51240 "
$N =$ 201 turns.

The mutual inductance was calculated from these dimensions by Professors Ayrton and Jones by the second formula of Jones given above. The result was as follows:

$$M = 2M_\theta = 18042.52 \text{ inches} = 45827.18 \text{ cm}$$

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[3] Proc. Roy. Soc., **63**, p. 196; 1898.

We may now calculate M for these two cases by formula (12) above.

	1st Case.	*2d Case.*
A	10.513365 inches.	10.512295 inches.
a	6.509985	6.307495
x	2.51240	2.51240
A^2	110.53087	110.50837
x^2	6.31215	6.31215
N	201.	201.
$d^2 = A^2 + x^2$	116.84302	116.82052
$\log d^2$	2.0676027	2.0675191
$\log \frac{a^2 A^2}{d^4} =$	$\bar{1}$.5354382	$\bar{1}$.5351848
$\log x^2/A^2 =$	$\bar{2}$.7566940	$\bar{2}$.7567824
X_2	2.77157	2.77152
X_4	1.94197	1.94185
X_6	1.2559	1.2557
X_8	0.6685	0.6683
X_{10}	0.184	0.184
1st term	1.0000000	1.0000000
2d "	.1286677	.1285926
3d "	.0254913	.0254611
4th "	.0053500	.0053526
5th "	.0010709	.0010682
6th "	.0001793	.0001787
7th "	.0000079	.0000078
Sum, $S =$	1.1607671	1.1606610
$\log S =$	0.0647451	0.0647054
" $2\pi^2 =$	1.2953298	1.2953298
" $a^2 =$	1.6271600	1.6268278
" $N\,(= 201) =$	2.3031961	2.3031961
	5.2904310	5.2900591
$\log d =$	1.0338013	1.0337595
$\log M =$	4.2566297	4.2562996
$\therefore M =$	18,056.34 inches.	18,042.62 inches.
Value by Jones's formula	18,056.36 "	18,042.52 "
Difference	−.02	+.10

The difference in the results obtained by formula (12) from the value obtained by Jones's formula amounts in the first case to about one part in a million, and in the second case to about five parts in a million. These differences are wholly negligible in the most refined experimental work. In using formula (12) one can judge the degree of approximation by the convergence of the terms, and so tell when enough terms have been calculated for the particular case. In the above example the breadth of the coil $2x$ is exceptionally small, and formula (12) is not as convergent as for wider coils.

EXAMPLE 2.

Take as a second example the case given by Jones[4] to illustrate his first formula.

$A = 10$ inches $\quad a = 5$ inches $\quad x = 2$ inches

$d^2 = 104 \qquad \frac{a^2 A^2}{d^4} = \frac{2500}{10816} \qquad \log \frac{a^2 A^2}{d^4} = \bar{1}.3638733$

$X_2 = 2.8400$
$X_4 = 2.1064$
$X_6 = 1.5208$
$X_8 = 1.0173$
$X_{10} = 0.5815$

1st	term	= 1.0000000
2	"	= .0866771
3	"	= .0118537
4	"	= .0017781
5	"	= .0002670
6	"	= .0000379
7	"	= .0000060
	Sum	= 1.1006198

$\frac{2\pi^2 a^2}{d} = 48.38972$

$\therefore M = 53.25868\ N$, N being the number of turns of wire on the coil.

Jones gives $M = 53.25879\ N$.

The difference between these values is 2 parts in a million.

[4] Phil. Mag., **27**, p. 61; 1889. In this example, P_0 should be 0.654870 instead of .54870, as printed in Jones's article.

5. SECOND DERIVATION OF EQUATION 12.

Gray[5] gives an expression for the mutual kinetic energy of two solenoidal coils from which equation (12) can be derived. Gray's expression is general; the coils may or may not be concentric and their axes may be at any angle with one another. The most important cases in practice is when the coils are concentric, and their axes are either in the same straight line or at right angles. Making the coils concentric causes half the terms in Gray's series to disappear. Making them coaxial reduces the zonal harmonic factor of each term to unity. When the coils have their axes at right angles the mutual energy becomes zero, but its derivative gives the torque between the coils, and we thus obtain the formula for the Gray absolute electrodynamometer. Putting the current in each coil equal to unity the mutual energy becomes the mutual inductance M.

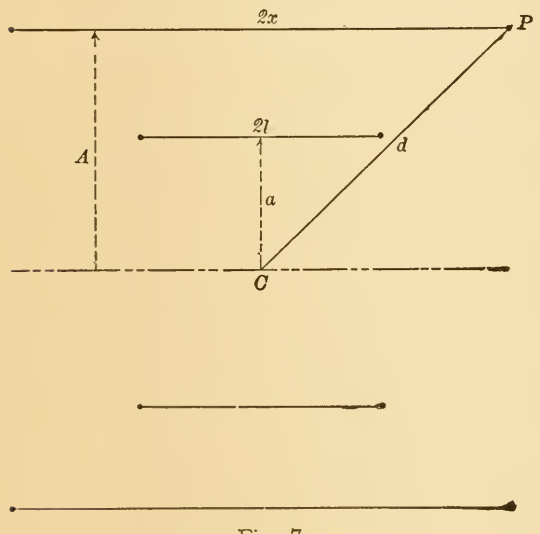

Fig. 7.

Put $2x$ = length of outer coil, Fig. 7.
$2l$ = length of inner coil, Fig. 7.
A = radius of outer coil, Fig. 7.
a = radius of inner coil, Fig. 7.
d = diagonal CP.
n_1, n_2 = number of turns per cm in the outer and inner windings respectively.
W = mutual kinetic energy.

With these changes Gray's expression for the mutual energy becomes:

$$W = \pi^2 a^2 A^2 n_1 n_2 i_1 i_2 \, (K_1 k_1 Z_1 + K_2 k_2 Z_2 + K_3 k_3 Z_3 + \ldots\ldots) \quad (14)$$

K_1, K_2, etc., are functions of the length and radius of the outer coil, k_1, k_2, etc., are functions of the length and radius of the inner coil, and Z_1, Z_2, etc., are zonal harmonic functions of the angle ϕ between the axes. When $\phi = 0$ the mutual inductance is

[5] Absolute Measurements, 2, Part I, p. 274.

$$M=\pi^2a^2A^2n_1n_2\,(K_1k_1+K_3k_3+K_5k_5+K_7k_7+K_9k_9+\ldots) \quad (15)$$

where

$$K_1=\frac{4x}{A^2d}$$

$$K_3=\frac{x}{d^5}$$

$$K_5=\frac{x}{4d^9}\left(4x^2-3A^2\right)$$

$$K_7=\frac{x}{4d^{13}}\left(4x^4-10x^2A^2+\frac{5}{2}A^4\right)$$

$$K_9=\frac{x}{4d^{17}}\left(4x^6-21x^4A^2+\frac{35}{2}x^2A^4-\frac{35}{16}A^6\right)$$

$$k_1=2l$$

$$k_3=l(4l^2-3a^2)$$

$$k_5=l\left(4l^4-10l^2a^2+\frac{5}{2}a^4\right)$$

$$k_7=l\left(4l^6-21l^4a^2+\frac{35}{2}l^2a^4-\frac{35}{16}a^6\right)$$

$$k_9=l\left(4l^8-36l^6a^2+63l^4a^4-\frac{105}{4}l^2a^6+\frac{63}{32}a^8\right)$$

Substituting these values in (15), and putting N_1 for $2xn_1$, and N_2 for $2ln_2$, N_1 and N_2 being the whole number of turns of wire on the outer and inner solenoids, respectively,

$$M=\frac{2\pi^2a^2N_1N_2}{d}\left\{1+\frac{A^2a^2}{8d^4}\left(3-4\frac{l^2}{a^2}\right)\right.$$
$$+\frac{A^4a^4}{32d^8}\left(3-4\frac{x^2}{A^2}\right)\left(\frac{5}{2}-10\frac{l^2}{a^2}+4\frac{l^4}{a^4}\right)$$
$$+\frac{A^6a^6}{32d^{12}}\left(\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}\right)\left(\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}+21\frac{l^4}{a^4}-4\frac{l^6}{a^6}\right)$$
$$+\frac{A^8a^8}{32d^{16}}\left(\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{l^4}{A^4}-4\frac{x^6}{A^6}\right)$$
$$\left.\left(\frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}+63\frac{l^4}{a^4}-36\frac{l^6}{a^6}+4\frac{l^8}{a^8}\right)\ldots\right\} \quad (16)$$

If we put $l=0$ and $N_2=1$, we have the case of the inner coil reduced to a circle at the center of the outer solenoid. In this case equation (16) reduces to (12) except that it is not carried out to

as many terms. Searle[6] and Airey have given an independent derivation of equation (16). Since deriving equation (12) I have found that Lorenz's equations (1) and (2) combined can be put into the same form, the four terms of (2) giving the first four terms of (12). I have verified the next two terms also by expanding the general term (2*a*). This operation is, however, somewhat tedious for the higher terms, as the *mth* term involves the *mth* derivative of $\left(\frac{a-1}{a}\right)^{m+\frac{1}{2}}$

6. EFFECT OF THICKNESS OF DISK.

The disk of a Lorenz apparatus is several millimeters in thickness at its edge, and it is important to know whether in any given case the mutual inductance is appreciably less than it would be for the ideal case of a disk of infinitesimal thickness assumed in all the formulæ. We can calculate the effect of the thickness by means of formula (16). Still keeping $N_2=1$, let l have a value equal to half the thickness of the disk. This effect will be greater with short coils than with long ones, but in any case the change would appear mainly in the first two terms after the unity term. Suppose, for example, that the disk of the McGill, Lorenz apparatus were 5 mm thick. Then l would be 0.25 cm, a is about 16.5 cm and $l/a=1/66$, $l^2/a^2=1/4356$. Neglecting terms in l^4/a^4 and higher powers, equation (16) may be written, putting $N_2=1$,

$$
\begin{aligned}
M=\frac{2\pi^2a^2N_1}{d}\Bigg\{ 1 &+\frac{A^2a^2}{8d^4}\left(3-4\frac{l^2}{a^2}\right)\\
&+\frac{A^4a^4}{32d^8}\left(3-4\frac{x^2}{A^2}\right)\left(\frac{5}{2}-\frac{10l^2}{a^2}\right)\\
&+\frac{A^6a^6}{32d^{12}}\left(\frac{5}{2}-10\frac{x^2}{A^2}+4\frac{x^4}{A^4}\right)\left(\frac{35}{16}-\frac{35}{2}\frac{l^2}{a^2}\right)\\
&+\frac{A^8a^8}{32d^{16}}\left(\frac{35}{16}-\frac{35}{2}\frac{x^2}{A^2}+21\frac{x^4}{A^4}-4\frac{x^6}{A^6}\right)\left(\frac{63}{32}-\frac{105}{4}\frac{l^2}{a^2}\right)+\cdots\Bigg\}
\end{aligned}
\qquad (17)
$$

Evidently each term after the first is made a little smaller by reason of the finite value of l. If we multiply the relative change of each term by the value of the term given on page 218, we shall have the change in M due to the thickness of the disk.

[6] The Electrician; Dec. 8, 1905.

Thus, since $l^2/a^2 = .000230$,

$.000230 \times \frac{4}{3}$ $= .000306$ = relative change in second term.
$.000230 \times 4$ $= .000920$ = relative change in third term.
$.000230 \times 8$ $= .001840$ = relative change in fourth term.
$.000230 \times 13\frac{1}{3}$ $= .003067$ = relative change in fifth term.
$.1287 \times .000306 = .0000394$ = change in S due to second term.
$.0255 \times .000920 = .0000235$ = change in S due to third term.
$.0053 \times .001840 = .0000097$ = change in S due to fourth term.
$.0011 \times .003067 = .0000034$ = change in S due to fifth term.

Total corrections $= .0000760 =$ 1 part in 15000 of M.

This reduction of the mean value of the mutual inductance of the coil and disk, due to the thickness of the disk, is small but very appreciable. If the thickness at the edge were reduced to 3 mm the change would be only .000027, or 1 in 43000, a quantity scarcely to be neglected, however, in the most exact work. As already suggested, this effect of the thickness of the disk would be reduced by making the coil longer, which is advantageous for other reasons. A longer coil will give a larger value of M, that is, a stronger field, which is very important, and also will reduce the variation in M due to the disk not being exactly centered. Displacement of the disk from the exact center along the axis reduces M; displacements of its center along the radius of the coil increases M. Obviously, such displacement would make no change in M if the coil were infinitely long and the field therefore uniform. But the longer the coil the more nearly uniform is the field, and hence the less is the error due to lack of exact centering.

The magnetic force at the center of the disk is $\frac{2\pi N_1}{d}$ the area of the disk is πa^2, hence $\frac{2\pi^2 a^2 N_1}{d}$ is the whole number of lines of force which would pass through the disk if the field were uniform and had throughout the value it has at the center. This is the first term in the expression for M, equation (12). Hence the values of the additional terms are a measure of the variation of the

field radially. For example, in the McGill apparatus, M is 16 per cent greater than it would be if the field were uniform and had the value it has at the center. If the coil were shorter this excess would be greater; if it were longer it would be less, and this is of course desirable.

7. EQUATION FOR THE GRAY ELECTRODYNAMOMETER.

When the axes of the two coils are at right angles to one another their mutual inductance is zero; but the moment of the force T tending to turn either coil about the center in the plane of their axes is a maximum and equal to $dW/d\phi$. Gray gives the general equation for this torque, which in the case of concentric coils becomes

$$T=\pi^2a^2A^2n_1n_2i_1i_2 \sin\phi\,(K_1k_1Z'_1+K_3k_3Z'_3+K_5k_5Z'_5+\ldots) \quad (18)$$

where ϕ is the angle between the axes of the coils. When $\phi=90°$, Z'_1, Z'_3, etc., have the following values:

$$Z'_1=1 \qquad Z'_7=-\frac{35}{16}$$

$$Z'_3=-\frac{3}{2} \qquad Z'_9=+\frac{315}{128}$$

$$Z'_5=+\frac{15}{8} \qquad Z'_{11}=-\frac{693}{256}$$

Substituting these values in (18) as well as the values of K_1, K_3, ... k_1, k_3..... given above, we obtain for the maximum value of the torque, for $\phi=90°$,

$$T_m=\frac{2\pi^2a^2N_1N_2i_1i_2}{d}\left\{1+\frac{3}{16}\frac{A^2a^2}{d^4}L_2-\frac{15}{256}\frac{A^4a^4}{d^8}X_2L_4+\frac{35}{512}\frac{A^6a^6}{d^{12}}X_4L_6\right.$$

$$\left.-\frac{315}{4096}\frac{A^8a^8}{d^{16}}X_6L_8+\ldots..\right\} \quad (19)$$

In this equation X_2, X_4, X_6, X_8 are functions of x and A and have the values given above (p. 216), and L_2, L_4, L_6, etc., are the same functions of l and a.

If the coils are so proportioned that the length of the winding is to the radius of the coil as $\sqrt{3}$ is to 1, for each coil, i. e., $4l^2=3a^2$ and $4x^2=3A^2$, L_2 and X_2 are zero and the expression for the torque becomes

$$T_m=\frac{2\pi^2a^2N_1N_2i_1i_2}{d}\left\{1+\frac{35}{512}\frac{A^6a^6}{d^{12}}X_4L_6-\frac{315}{4096}\frac{A^8a^8}{d^{16}}X_6L_8+\ldots\right\} \quad (20)$$

But in this case, since $d^2=\frac{7}{4}A^2$,

$$\frac{A^6a^6}{d^{12}}=\left(\frac{4}{7}\right)^{12}\frac{a^6}{A^6} \qquad X_4=-\frac{11}{4}$$

$$\frac{A^8a^8}{d^{16}}=\left(\frac{4}{7}\right)^{16}\frac{a^8}{A^8} \qquad X_6=L_6=-\frac{13}{16}$$

$$X_8=L_8=+\frac{243}{64}$$

$$\therefore\ T_m=\frac{2\pi^2a^2N_1N_2i_1i_2}{d}\left\{1+.0001851\frac{a^6}{A^6}+.0000307\frac{a^8}{A^8}+\ldots\right\} \quad (21)$$

If $a=A/2$, the correction terms together amount to only a few parts in a million and may be neglected. Hence, if the relations $4x^2=3A^2$ and $4l^2=3^2a$ are exactly realized, the first term is enough to use even in the most refined experimental work and with the largest moving coil that it is practicable to use. If, however, this relation is not quite realized (as it probably never would be exactly) the slight correction to be made can be calculated from the second and third terms of (19).

8. THE AYRTON-JONES ELECTRODYNAMOMETER.

The Ayrton-Jones absolute electrodynamometer[7] consists essentially of a cylinder wound with a single layer of wire suspended from the arm of a balance inside of and coaxial with a larger fixed cylinder, the latter being also wound with a single layer of wire,

[7] Journal Inst. E. E., **35**, p. 11; 1904-5.

Fig. 8. The winding of the fixed cylinder is, however, divided into two parts, the current flowing in opposite directions in the two halves. Thus, if the effect of the current in the lower half BC is to draw the inner cylinder ED down, the upper half of the cylinder AB will (because the current flows in opposite direction) repel the suspended cylinder and so urge it down, with a force equal to the force of BC, if the inner cylinder is suspended symmetrically with respect to the other two. Since the whole force is twice the force acting between AB and ED, we may now disregard the lower cylinder in finding the value of the force, simply doubling the force calculated for AB on ED. Jones[8] has shown that the force

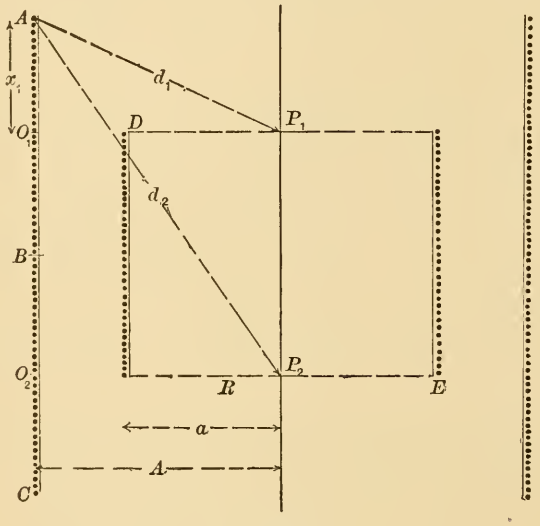

Fig. 8.

between AB and ED is proportional to the difference in the mutual inductances M_1 and M_2 of the coil AB on the two circles S and R at the ends of the inner cylinder. If the currents in the two coils are i_1, i_2, and n_2 is the number of turns of wire per cm on ED, the force in dynes on ED is

$$F = i_1 i_2 n_2 (M_1 - M_2)$$

Professor Jones gave two proofs of this formula, the second[9] of which, because of its simplicity and importance, I shall reproduce here.

---

[8] Proc. Roy. Soc., **63**, p. 198; 1898.

[9] This proof was suggested to Professor Jones by Prof. Andrew Gray.

If dx is an element of the suspended coil at the end S, Fig. 10, and M_1 is the mutual inductance of a circle at S with respect to the outer coil, the potential energy of this element is

$$W_1 = i_1 i_2 M_1 n_2 dx,$$

where i_1 and i_2 are the currents in the two coils, respectively, n_2 is the number of turns of wire per cm on the suspended coil, and dx is the length of the element. $M_1 i_1$ is the energy of unit current in one turn at S; the number of current turns is $n_2 dx i_2$. If this element be at R its potential energy is

$$W_2 = i_1 i_2 M_2 n_2 dx,$$

where M_2 is the mutual inductance of a circle at R with respect to the outer winding. If the element be carried from S to R, work is done equal to

$$W_1 - W_2 = i_1 i_2 (M_1 - M_2) n_2 dx.$$

This is, however, equivalent to moving the whole cylinder vertically through a distance dx. Therefore, if F be the force acting on the suspended cylinder,

$$F dx = i_1 i_2 n_2 (M_1 - M_2) dx$$

or

$$F = i_1 i_2 n_2 (M_1 - M_2), \qquad \text{as stated above.}$$

It is hence only necessary to calculate M_1 and M_2 by formula (1), (3), or (12) in order to be able to calculate the force due to unit current in the balance; or, conversely, knowing the force F by weighing, the absolute value of the current flowing in the coils can be calculated.

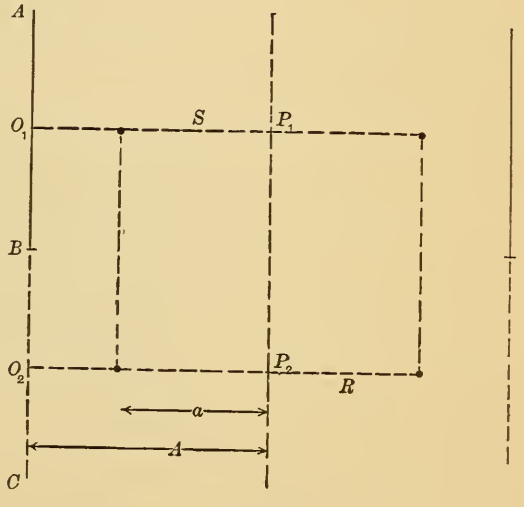

Fig. 9.

If the inner coil has the same length as the coil AB and is symmetrically situated with respect to the two coils, then the two end circles S and R will lie in planes passing through the middle of AB and BC, respectively.

The mutual inductance of AB on S, Fig. 9, is twice the mutual inductance of AO_1 on S. Similarly, the mutual inductance of AB on R is the mutual inductance of AO_2 on the circle R minus that of BO_2 on R. Hence the same formula can be used in obtaining M_2 that is used for M_1, merely varying the value of x, the distance from the plane of the circle to the end of the winding and of the diagonal d; that is, the three values of x will be O_1A, O_2A, and O_2B in the three cases, respectively, and the corresponding values of d will be D_1A, D_2A, and D_2B.

9. CALCULATION OF ELECTRODYNAMOMETER CONSTANT.

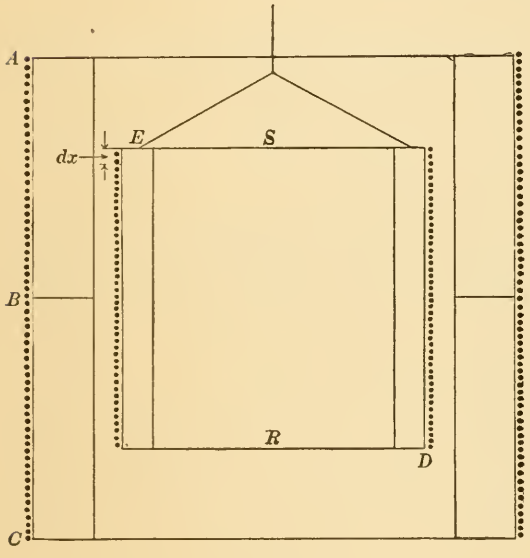

Fig. 10.

As a further test of the formulæ let us calculate the constant of an electrodynamometer of the Ayrton-Jones type, of which AB, Fig. 8, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of tne lower fixed coil BC.

Assume the dimensions as follows:

A = 16 cm = radius of fixed coil, Fig. 10.
a = 10 cm = radius of moving coil.
x_1 = 8 cm = half length of AB = O_1A
x_2 = 24 cm = 1.5 times AB = O_2A
n_1 = 10 = number of turns per cm
N_1 = 80 = number of turns in distance $O_1A = x_1$, Fig. 9.
N_2 = 240 = number of turns in distance $O_2A = x_2$

$$d_1 = \sqrt{A^2 + x_1^2} = 8\sqrt{5} = \text{diagonal } AP_1, \text{ Fig. 9.}$$

$$d_2 = \sqrt{A^2 + x_2^2} = 8\sqrt{13} = \text{diagonal } AP_2$$

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil

O_2A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (12) will be used, taking N_1, x_1, and d_1 in the first case and N_2, x_2, and d_2 in the second case.

	For M_S	*For* M_R
A	16 cm	16 cm
a	10	10
x	8	24
A^2	256	256
x^2	64	576
$N=nx$	80	240
d^2	320	832
log d^2	2.5051500	2.9201233
$\frac{a^2A^2}{d^4}$	$\bar{1}$.3979400	$\bar{2}$.5679934
$\frac{x^2}{A^2}$	$\bar{1}$.3979400	0.1760913
X_2	+2.000	− 6.00
X_4	+0.250	+ 0.25
X_6	−0.9375	+23.5
X_8	−1.203	−45.7
X_{10}	−0.562	−49.0
1st term	1.0000000	1.0000000
2d "	+ .0937500	+ .0138683
3d "	+ .0097656	− .0006411
4th "	+ .0002670	+ .0000009
5th "	− .0002253	+ .0000027
6th "	− .0000662	− .0000002
7th "	− .0000036	.0000000
Sum $=S$	1.1034875	1.0132306
log S_1 =	0.0427674	log S_2 = 0.0057083
" $2\pi^2$ =	1.2953298	" $2\pi^2$ = 1.2953298
" $a^2(=100)$ =	2.0000000	" $a^2(=100)$ = 2.0000000
" $N_1(=80)$ =	1.9030900	" $N_2(=240)$ = 2.3802112
	5.2411872	5.6812493
" d_1 =	1.2525750	" d_2 = 1.4600616
log M_S =	3.9886122	log M_R = 4.2211877
$\therefore M_S$ =	9741.19	M_R = 16641.32

10. EXAMPLE 2 BY JONES'S FORMULA.

We will now calculate M_S and M_R by Jones's second formula given above, using also the following equation to find $F-\Pi$:

$$\frac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c} = F(k)E(k',\beta)+E(k)F(k',\beta)-F(k)F(k',\beta)-\frac{\pi}{2}$$

	For M_S	*For* M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta = 2\pi N$	160 π	480 π
$c=\frac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c'=\sqrt{1-c^2}$	0.2307692	0.2307692
$k=\frac{2\sqrt{Aa}}{\sqrt{(A+a)^2+x^2}}$	0.9299812	0.7149701
$k'=\sqrt{1-k^2}$	0.3676073	0.6991550
$\log \sin\beta \left(\sin\beta=\frac{c'}{k'}\right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\frac{F-E}{k^2}$	1.5088957	1.0146546
$F(k', \beta)$	0.6852557	0.3394833
$E(k', \beta)$	0.6721988	0.3333201
$\frac{k'^2 \sin\beta\cos\beta(F-\Pi)}{c}$	−0.8266738	−1.1256799
$\frac{c'^2}{c^2}(F-\Pi)$	−0.6851799	−0.4045298
$\log\left\{\frac{F-E}{k^2}+\frac{c'^2}{2}(F-\Pi)\right\}$	$\bar{1}$.9157773	$\bar{1}$.7854187
$\log(\Theta(A+a)ck)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_S = 9741.17$ cm	$M_R = 16641.32$

M_S differs from the value obtained by formula (12) by 2 parts in a million, M_R is identical.

M_S is the mutual inductance of the winding O_1A on S. The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1 = 19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S, that is, M_S. Therefore.

$$M_2 = 16641.32 - 9741.17 = 6900.15$$

Hence $M_1 - M_2 = 12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\tfrac{1}{2}f = i_1 i_2 n_2 (M_1 - M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1 = i_2 = 1$ ampere $= 0.1$ c.g.s. unit of current and $n_2 = 10$ turns per cm. Then

$$i_1 i_2 n_2 = 0.10$$

$$\therefore f = 0.20 \times 12582.19 \text{ dynes}$$
$$= 2516.438 \text{ dynes}$$
$$2f = 5032.876 \text{ dynes} = \text{change of force on reversal of current}$$
$$= 5.1356 \text{ gms where } g = 980.$$

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (12) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

11. EQUIVALENCE OF SPIRAL WINDING AND CURRENT SHEET.

The derivation of Lorenz's formula is given in his collected works.[10] Its exact equivalence to equation (12) might have been anticipated from the fact that both are based on the same hypothesis, namely, that the current is uniformly distributed over the solenoid in a cur-

[10] Oeuvres Scientifiques de L. Lorenz, t. 2—1. Copenhague, 1899.

rent sheet, as is Jones's first formula. Jones's second formula, on the other hand, assumes the current flowing along the axis of a wire wound spirally around the solenoid, making some integral number of turns but having any pitch whatever. The exact equivalence of this formula to (1) and (12) is at first surprising, but Lord Rayleigh has shown[11] from simple physical considerations that this must be true when as in this case the disk is circular and coaxial with the solenoid.

In an actual instrument, however, the current flows neither in a current sheet nor along the *axis* of a spiral wire, but is distributed

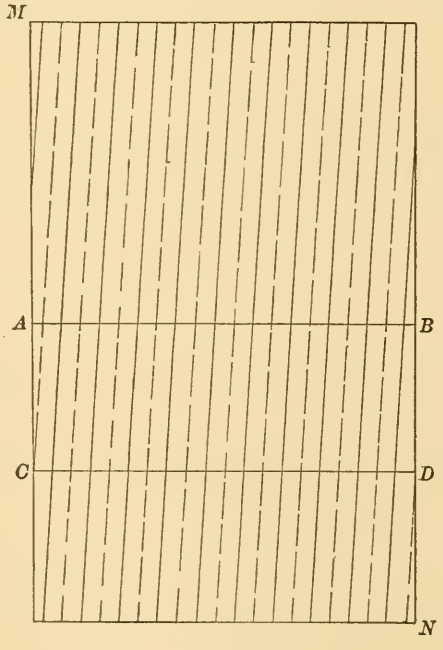

Fig. 11.

throughout the entire section of the spirally wound wire. It is therefore worth while to inquire whether the departure of an absolute electro-dynamometer from the ideal conditions assumed in the theory of the instrument can be the source of an appreciable error. The case is similar to the one I have considered elsewhere[12] in which the winding was assumed circular. It was shown that no appreciable error is due to the finite cross section of the wire. Let us now consider a spiral of fine wire wound on a short cylinder. Let MN, Fig. 11, be the surface of the cylinder developed in a plane, and AB a

[11] Report Brit. Asso., p. 241; 1899. [12] This Bulletin, 2, p. 71; 1906.

winding of n turns of uniform pitch, beginning and ending in the two bounding planes of the cylinder. The magnetic force on the axis at the center of the cylinder due to unit current in this wire is independent of the particular points A, B, where the winding begins and ends. A second winding C, D, indicated by the dotted line, will produce exactly the same magnetic force at the center as AB, or exactly the same number of lines of force through any circle coaxial with the cylinder. Hence it is immaterial whether the current flows through one of these wires or through both in parallel or through any number of similar windings all in parallel, the total

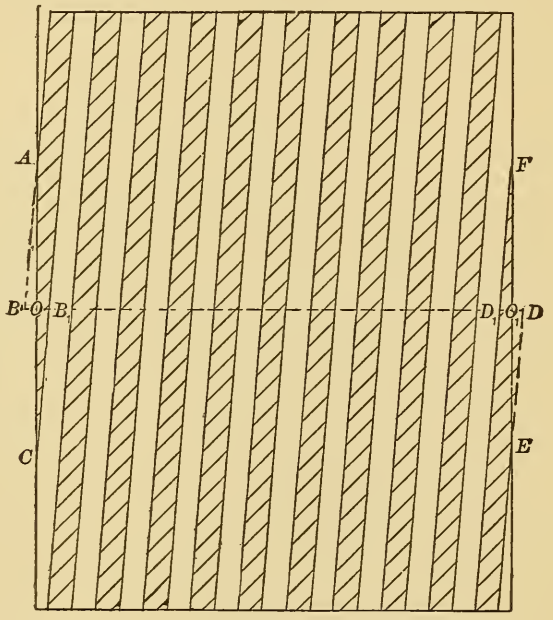

Fig. 12.

current being of course the same in each case. Hence a winding of flat tape, Fig. 12, of any width having the same pitch and number of turns will have the same mutual inductance with respect to any coaxial circle as a winding of very fine wire, *provided the current begins and ends in the bounding planes of the cylinder.* If the distance between the separate turns of tape is reduced to zero we have a current sheet, which is thus seen to be equivalent to a winding of fine wire.

If the tape be wound on edge, Fig. 11, the winding may begin at any point in the bounding plane M provided it ends at a corresponding point in the other bounding plane N. By the same reason-

ing as before a winding of n turns of thin tape on edge has the same magnetic force at any point on the axis and the same mutual inductance with respect to any coaxial circle as a winding of square or rectangular wire, the depth of the latter being equal to the depth of the tape. When the breadth of the rectangular wire is equal to the pitch of the winding, the wire covers the entire surface and we have a thick current sheet. It can easily be proved[13] that for the dimensions of wire and cylinder likely to be employed in any absolute electrodynamometer, the effect of a thick current sheet equivalent to the winding will not be appreciably different from that of a thin current sheet having a radius equal to the mean radius of the thick sheet.

12. EFFECT OF LEAD WIRES.

We have assumed in what precedes that the current enters and leaves the winding in the bounding planes MN of the short cylinder on which the wire is wound. Thus, if the winding consists of thin or thick tape, the breadth of which equals half the pitch, the bounding planes will cut off the conductor diagonally, leaving the wedge-shaped terminals AC and EF (Fig. 12), and the mathematical conditions would require that the current be introduced along the entire distance CA and leave along EF, so that the magnitude of the current at any point would be proportional to the breadth of the tape. As CA is half the circumference of the cylinder, this evidently is an impracticable method of introducing the current. Let us therefore inquire whether it will make any appreciable difference if the conductor is cut off abruptly at BB_1, DD_1 by a plane passing through the axis of the cylinder, and the current be introduced at BB_1 and withdrawn at DD_1. We have the triangular portion of the conductor COB_1 omitted and in its place the portion AOB; likewise EO_1D replaces FO_1D_1. The current in COB_1 averages one-fourth of the total current and the distance CO is one-fourth of one circumference. Hence the magnetic effect of COB_1 is practically equivalent to one-sixteenth of one turn at the end, or one-twenty-eighth of the average turn in the Gray electrodynamometer, such as that discussed in the article to which reference was

[13] This Bulletin, 2, p. 71; 1906.

made above. That instrument had altogether 872 turns, and hence the effect of the triangular portion of the terminal COB_1 is about one-twenty-five thousandth of the total. This is practically negligible, seeing that it would produce an error of only 1 in 50000 in the current. But it is compensated almost exactly by the portion ABO, which is a little further from the center of the coil than COB_1. In the dynamometer in question there were 20 turns per centimeter, and hence BB_1 = one-fourtieth cm. The difference in the average distances of the two elements from the central plane of the instrument would therefore be less than one-eightieth cm, and the difference in their magnetic effects would therefore be less than one-one thousand five hundredth of either. Hence the error produced by introducing the total current at the end BB_1 of the wire or tape would be less than $\frac{1}{1500} \times \frac{1}{25000}$, or less than 1 part in 37,500,000, or for the two ends together 1 in 18,750,000. If the coil had a smaller radius or coarser winding the error would be greater; but it probably never would amount to as much as one part in a million for any instrument designed to be used as an absolute instrument.

We can therefore be sure that the formulæ derived for the Gray, Ayrton-Jones, and other electrodynamometers having large coils, or for the Lorenz apparatus, on the assumption that the current is distributed over the surface of the cylinder as a current sheet can be safely employed provided the leads are properly twisted together and the return current is brought from DD_1, back to BB_1 along an element of the cylinder. The length of cylinder to be employed in the formulæ is, as already explained[14], *the over-all length of the winding including the insulation on the first and last wires*, when the winding consists of a single layer of insulated wire wound with the adjacent turns in contact. For a winding *in a screw thread it is n times the pitch*, or *the length from center to center of n+1 turns*, *n* being the whole number of turns of wire on the cylinder.

I have here discussed these questions of the equivalence of a winding of wire to a current sheet and of the length of the equivalent current sheet, because I have recently received letters of inquiry from persons engaged in absolute measurements, who were not

[14] This Bulletin, 2, p. 77; 1906.

quite satisfied that the conclusion reached in my former paper applied to a spiral winding. I hope that the above discussion makes it clear that it does. A spiral winding approaches a circular winding as the pitch decreases, and nothing is assumed in the preceding as to the magnitude of the pitch.

WASHINGTON, March 1, 1907.

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